

Matrix Polynomials with Specified Eigenvalues

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Abstract

This work concerns the distance in 2-norm from a matrix polynomial to a nearest polynomial with a specified number of its eigenvalues at specified locations in the complex plane. Perturbations are allowed only on the constant coefficient matrix. Singular value optimization formulas are derived for these distances facilitating their computation. The singular value optimization problems, when the number of specified eigenvalues is small, can be solved numerically by exploiting the Lipschitzness and piece-wise analyticity of the singular values with respect to the parameters.

Key words. Matrix Polynomial, Linearization, Singular Values, Sylvester Equation, Eigenvalue Perturbation

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1 Introduction

We study the distance from a matrix polynomial to the nearest polynomial with specified number of eigenvalues at specified positions in the complex plane. Formally, let $P : \mathbb{C} \rightarrow \mathbb{C}^{n \times n}$, defined by

$$P(\lambda) := \sum_{j=0}^m \lambda^j A_j, \quad (1)$$

be a square matrix polynomial where $A_j \in \mathbb{C}^{n \times n}$. Throughout the paper we will assume that $\text{rank}(A_m) = n$. Suppose also that a set $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ consisting of complex scalars and $r \in \mathbb{Z}^+$ are given. This paper provides a singular value formula for the distance

$$\beta(P, \mathcal{S}) := \inf \left\{ \|\Delta A\|_2 \mid \Delta A \in \mathbb{C}^{n \times n} \text{ s.t. } \sum_{j=1}^s m_j (P(\lambda_j) + \Delta A) \geq r \right\} \quad (2)$$

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where $m_j(P)$ denotes the algebraic multiplicity of λ_j as an eigenvalue of $P(\lambda)$, that is the multiplicity of λ_j as a root of the polynomial $\det(P(\lambda))$.

The formula derived is a generalization of the singular value characterization in [3] for a linear matrix pencil of the form $L(\lambda) = A_0 + \lambda A_1$, which was inspired by the Malyshev's work [4] earlier. However, unlike [4] the derivation here fully depends on a Sylvester equation characterization for a matrix polynomial to have sufficiently many eigenvalues belonging to \mathcal{S} . This yields a neater derivation. The distance from a matrix polynomial to the nearest one with a multiple eigenvalue was considered in [5], where singular value formulas yielding only lower and upper bounds were derived. Even though in [5] perturbations to all of the coefficient matrices were allowed, it is not clear how tight the derived bounds are. Here the derived singular value formula, when the number of prescribed eigenvalues in \mathcal{S} is small, facilitates the numerical computation of the distances by means of the algorithms exploiting the Lipschitzness [6, 7] and piece-wise analyticity of singular values [2].

In the next section we provide a Sylvester equation and a rank characterization for the condition $\sum_{j=1}^s m_j(P) \geq r$. The rank characterization provides a singular value formula bounding the actual distance from below right away. In section 3 we establish the exact equality of the singular value formula with the distance by constructing an optimal perturbation. The derived singular value formula (Theorem 3.1) and its corollaries are summarized in subsection 3.3. Section 4 illustrates the validity of the results in practice on two examples and making connections with the ϵ -pseudospectrum for a matrix polynomial.

Notation: Throughout the text \mathcal{S}^r denotes the r tuples with elements from the set \mathcal{S} , $\sigma_k(\cdot)$ denotes the k th largest singular value of its matrix argument, and \otimes denotes the Kronecker product.

2 Rank characterization for polynomials with specified eigenvalues

We first deduce a rank characterization, for a given set of complex scalars $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ and a positive integer r , that confirms whether the scalars in \mathcal{S} are eigenvalues of the polynomial $P(\lambda)$ as defined in (1) with algebraic multiplicities summing up to r or greater. Formally, if we let $m_j(P)$ denote the algebraic multiplicity of λ_j as an eigenvalue of $P(\lambda)$, that is the multiplicity of λ_j as a root of $\det(P(\lambda))$, then we are seeking a rank characterization for the condition

$$\sum_{j=1}^s m_j(P) \geq r.$$

The derivation exploits the linearization $\mathcal{L}(\lambda) := \mathcal{A} + \lambda \mathcal{B}$ for $P(\lambda)$ with

$$\mathcal{A} := \begin{bmatrix} 0 & I & & 0 \\ & & \ddots & \\ 0 & 0 & & I \\ A_0 & A_1 & & A_{m-1} \end{bmatrix} \quad \text{and} \quad \mathcal{B} := \begin{bmatrix} -I & 0 & 0 \\ & \ddots & \\ 0 & -I & 0 \\ 0 & 0 & A_m \end{bmatrix}, \quad (3)$$

and benefits from the fact that the eigenvalues of $L(\lambda)$ and $P(\lambda)$ are the same with the same algebraic multiplicities. Due to the assumption that $\text{rank}(A_m) = n$ the matrix \mathcal{B} is full rank.

Consequently, we could apply Theorem 2.1 concerning the multiplicities of the eigenvalues of matrix pencils given below to the pencil $\mathcal{L}(\lambda)$. The result originally appeared in [3, Theorem 3.3]. For the theorem we introduce the notation

$$C(\mu, \Gamma) = \begin{bmatrix} \mu_1 & -\gamma_{21} & \cdots & -\gamma_{r1} \\ 0 & \mu_2 & \ddots & \vdots \\ & & \ddots & -\gamma_{r(r-1)} \\ 0 & & & \mu_r \end{bmatrix}, \quad (4)$$

where

$$\mu = [\mu_1 \ \mu_2 \ \cdots \ \mu_r]^T \in \mathcal{S}^r \quad \text{and} \quad \Gamma = [\gamma_{21} \ \gamma_{31} \ \cdots \ \gamma_{r,r-1}]^T \in \mathbb{C}^{r(r-1)/2}.$$

We also denote the generic set of Γ values such that $C(\mu, \Gamma)$ has all eigenvalues with geometric multiplicities equal to one by $\mathcal{G}(\mu)$.

Theorem 2.1. *Let $L(\lambda) := A + \lambda B$ be a matrix pencil with $A, B \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(B) = n$, $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ be a set of complex scalars, and $r \in \mathbb{Z}^+$. The following two conditions are equivalent.*

(1) $\sum_{j=1}^s m_j(A, B) \geq r$ where $m_j(A, B)$ is the algebraic multiplicity of λ_j as an eigenvalue $L(\lambda) = A + \lambda B$.

(2) There exists a $\mu \in \mathcal{S}^r$ such that for all $\Gamma \in \mathcal{G}(\mu)$

$$\dim \{X \in \mathbb{C}^{n \times r} \mid AX + BXC(\mu, \Gamma) = 0\} \geq r.$$

Theorem 2.2. *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$, $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ be a set of complex scalars, and $r \in \mathbb{Z}^+$. The following two conditions are equivalent.*

(1) $\sum_{j=1}^s m_j(P) \geq r$ where $m_j(P)$ is the algebraic multiplicity of λ_j as an eigenvalue $P(\lambda)$.

(2) There exists a $\mu \in \mathcal{S}^r$ such that for all $\Gamma \in \mathcal{G}(\mu)$

$$\dim \left\{ X \in \mathbb{C}^{n \times r} \mid \sum_{j=0}^m A_j X C^j(\mu, \Gamma) = 0 \right\} \geq r.$$

Proof. We apply Theorem 2.1 to the linearization (3) for $P(\mu)$. It follows from Theorem 2.1 that the condition $\sum_{j=1}^s m_j(P) \geq r$ is met if and only if

$$\dim \{X \in \mathbb{C}^{mn \times r} \mid \mathcal{A}X + \mathcal{B}XC(\mu, \Gamma) = 0\} \geq r.$$

On the other hand the partitioning $X = [X_0^T \ X_1^T \ \cdots \ X_{m-1}^T]^T$ where $X_j \in \mathbb{C}^{n \times r}$ reveals that the condition

$$0 = \mathcal{A}X + \mathcal{B}XC(\mu, \Gamma) = \begin{bmatrix} X_1 \\ \vdots \\ X_{m-1} \\ \sum_{j=0}^{m-1} A_j X_j \end{bmatrix} + \begin{bmatrix} -X_0 C(\mu, \Gamma) \\ \vdots \\ -X_{m-2} C(\mu, \Gamma) \\ A_m X_{m-1} C(\mu, \Gamma) \end{bmatrix}$$

could be expressed as $X_j = X_{j-1}C(\mu, \Gamma)$ for $j = 1, \dots, m-1$ and

$$\sum_{j=0}^{m-1} A_j X_j + A_m X_{m-1} C(\mu, \Gamma) = 0.$$

By eliminating X_j for $j = 0, \dots, m-1$ in the last equation using $X_j = X_0 C(\mu, \Gamma)^j$ we obtain

$$\sum_{j=0}^m A_j X_0 C(\mu, \Gamma)^j = 0.$$

To summarize X_0 is a solution of $\sum_{j=0}^m A_j X C(\mu, \Gamma)^j = 0$ if and only if

$$\mathcal{X}_0 = \begin{bmatrix} X_0^T & (X_0 C(\mu, \Gamma))^T & \dots & (X_0 C^{m-1}(\mu, \Gamma))^T \end{bmatrix}^T$$

is a solution of $\mathcal{A}\mathcal{X} + \mathcal{B}\mathcal{X}C(\mu, \Gamma) = 0$ and the result follows. \square

Finally we express the last Sylvester characterization as a rank condition.

Corollary 2.3. *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$, $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ be a set of complex scalars, and $r \in \mathbb{Z}^+$. The following two conditions are equivalent.*

- (1) $\sum_{j=1}^s m_j(P) \geq r$ where $m_j(P)$ is the algebraic multiplicity of λ_j as an eigenvalue $P(\lambda)$.
- (2) There exists a $\mu \in \mathcal{S}^r$ such that for all $\Gamma \in \mathcal{G}(\mu)$

$$\text{rank} \left(\sum_{j=0}^m (C^j(\mu, \Gamma))^T \otimes A_j \right) \leq n \cdot r - r.$$

Proof. The result follows from Theorem 2.2 by applying the identity

$$\text{vec}(AXB) = (B^T \otimes A) \text{vec}(X)$$

to the Sylvester equation $\sum_{j=0}^m A_j X C^j(\mu, \Gamma) = 0$ \square

For instance we deduce the following when $\mathcal{S} = \{\mu\}$ and $r = 2$ from the corollary above; the matrix polynomial $P(\lambda)$ has μ as a multiple eigenvalue if and only if

$$\text{rank} \left(\sum_{j=0}^m \begin{bmatrix} \mu & 0 \\ \gamma & \mu \end{bmatrix}^j \otimes A_j \right) = \text{rank} \left(\begin{bmatrix} P(\mu) & 0 \\ \gamma P'(\mu) & P(\mu) \end{bmatrix} \right) \leq 2n - 2$$

for all $\gamma \neq 0$.

3 Derivation of the Singular Value Formula

For each $\mu \in \mathcal{S}^r$ let us define the quantity

$$\beta(P, \mu) := \inf \left\{ \|\Delta A\|_2 \mid \text{rank} \left(\sum_{j=1}^m (C^j(\mu, \Gamma))^T \otimes A_j + I \otimes (A_0 + \Delta A) \right) \leq n \cdot r - r \right\}$$

for any $\Gamma \in \mathcal{G}(\mu)$. Then, from Corollary 2.3, the distance to the nearest polynomial with specified eigenvalues could be expressed as

$$\beta(P, \mathcal{S}) := \inf_{\mu \in \mathcal{S}^r} \beta(P, \mu),$$

so it suffices to derive a singular value formula for $\beta(P, \mu)$.

We immediately deduce the lower bound

$$\beta(P, \mu) \geq \kappa(P, \mu) := \sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \sigma_{n \cdot r - r + 1} \left(\sum_{j=1}^m (C^j(\mu, \Gamma))^T \otimes A_j + I \otimes A_0 \right), \quad (5)$$

since for any matrix B the distance in 2-norm to the nearest matrix of rank r is given by $\sigma_{r+1}(B)$. We could not immediately deduce the upper bound because the allowable perturbations have structure, specifically they are of the form $I \otimes \Delta A$.

To establish the validity of the reverse inequality $\beta(P, \mu) \leq \kappa(P, \mu)$ it is sufficient to construct a perturbation ΔA_* such that

(i) $\|\Delta A_*\|_2 = \kappa(P, \mu)$, and

(ii) $\text{rank} \left(\sum_{j=1}^m (C^j(\mu, \Gamma))^T \otimes A_j + I \otimes (A_0 + \Delta A_*) \right) \leq n \cdot r - r$ for some $\Gamma \in \mathcal{G}(\mu)$.

It can be shown that the singular value function in (5) approaches zero as any of the components of Γ goes to ∞ . (By using Cramer's rule it is possible to deduce that the n largest singular values of the inverse of the matrix in (5) blow up as one of the components of Γ goes to ∞ .) Since the singular value function depends on Γ continuously, the supremum in (5) must be attained. Let Γ_* be such a point where the supremum in (5) is attained, that is

$$\kappa(P, \mu) = \sigma_{n \cdot r - r + 1} \left(\sum_{j=1}^m (C^j(\mu, \Gamma_*))^T \otimes A_j + I \otimes A_0 \right). \quad (6)$$

Let $\mathcal{U}, \mathcal{V} \in \mathbb{C}^{nr}$ be a consistent pair of unit left and right singular vectors associated with this singular value, in particular \mathcal{U} and \mathcal{V} satisfy

$$\left(\sum_{j=1}^m (C^j(\mu, \Gamma_*))^T \otimes A_j + I \otimes A_0 \right) \mathcal{V} = \kappa(P, \mu) \mathcal{U} \quad (7)$$

and

$$\mathcal{U}^* \left(\sum_{j=1}^m (C^j(\mu, \Gamma_*))^T \otimes A_j + I \otimes A_0 \right) = \kappa(P, \mu) \mathcal{V}^*. \quad (8)$$

Throughout the rest of this section we prove that

$$\Delta A_* := -\kappa(P, \mu)UV^+ \quad (9)$$

satisfies both of the properties (i) and (ii) above, where $U, V \in \mathbb{C}^{n \times r}$ are such that $\mathcal{U} = \text{vec}(U)$ and $\mathcal{V} = \text{vec}(V)$ under the following mild assumptions.

1. **(Multiplicity Assumption)** The multiplicity of the singular value

$$\sigma_{n \cdot r - r + 1} \left(\sum_{j=1}^m (C^j(\mu, \Gamma_*))^T \otimes A_j + I \otimes A_0 \right)$$

is one.

2. **(Linear Independence Assumption)** $\text{rank}(V) = r$

3.1 Norm of ΔA_*

We aim to show that $\|\Delta A_*\|_2 = \kappa(P, \mu)$. For this purpose, it is sufficient to establish the validity of $U^*U = V^*V$, since this property implies

$$\|UV^+\|_2 = \max_{w \in \mathbb{C}^n, \|w\|_2=1} \sqrt{(V^+)^* U^* U V^+ w} = \max_{w \in \mathbb{C}^n, \|w\|_2=1} \sqrt{(V^+)^* V^* V V^+ w} = \|V V^+\|_2 = 1$$

where the last equality is due to the fact that $V V^+$ is an orthogonal projector.

Throughout the rest of this subsection we prove the property $U^*U = V^*V$ under the multiplicity assumption. Let

$$\sigma(\Gamma) := \sigma_{n \cdot r - r + 1}(F(\Gamma)), \quad \text{where} \quad F(\Gamma) := \sum_{j=1}^m (C^j(\mu, \Gamma))^T \otimes A_j + I \otimes A_0.$$

Then the partial derivatives of $F(\Gamma)$ with respect to the real and the imaginary parts of the components γ_{ik} of Γ are

$$\begin{aligned} \frac{\partial F}{\partial \Re \gamma_{ik}}(\Gamma) &= \sum_{j=1}^m \left(\sum_{\ell=0}^{j-1} C^\ell(\mu, \Gamma) \frac{\partial C(\mu, \Gamma)}{\partial \Re \gamma_{ik}} C^{j-1-\ell}(\mu, \Gamma) \right)^T \otimes A_j \\ &= - \sum_{j=1}^m \sum_{\ell=0}^{j-1} (C^\ell(\mu, \Gamma) e_i e_k^\top C^{j-1-\ell}(\mu, \Gamma))^T \otimes A_j, \\ \frac{\partial F}{\partial \Im \gamma_{ik}}(\Gamma) &= \sum_{j=1}^m \left(\sum_{\ell=0}^{j-1} C^\ell(\mu, \Gamma) \frac{\partial C(\mu, \Gamma)}{\partial \Im \gamma_{ik}} C^{j-1-\ell}(\mu, \Gamma) \right)^T \otimes A_j \\ &= -i \sum_{j=1}^m \sum_{\ell=0}^{j-1} (C^\ell(\mu, \Gamma) e_i e_k^\top C^{j-1-\ell}(\mu, \Gamma))^T \otimes A_j, \end{aligned}$$

where e_i (e_k) denotes the i th (the k th) column of the $r \times r$ unit matrix, and $1 \leq i < k \leq r$. Let

$$G := \sum_{j=1}^m \sum_{\ell=0}^{j-1} C^{j-1-\ell}(\mu, \Gamma_*) U^* A_j V C^\ell(\mu, \Gamma_*). \quad (10)$$

From the assumption that the singular value $\sigma(\Gamma_*)$ is simple it follows that the function $\Gamma \mapsto \sigma(\Gamma)$ is analytic at Γ_* , and

$$\begin{aligned} 0 &= \frac{\partial \sigma}{\partial \Re \gamma_{ik}}(\Gamma_*) \\ &= \Re \left(U^* \frac{\partial F}{\partial \Re \gamma_{ik}}(\Gamma_*) V \right) \\ &= \Re \left(\text{vec}(U)^* \frac{\partial F}{\partial \Re \gamma_{ik}}(\Gamma_*) \text{vec}(V) \right) \\ &= -\Re \left(\text{vec}(U)^* \text{vec} \left(\sum_{j=1}^m \sum_{\ell=0}^{j-1} A_j V C^\ell(\mu, \Gamma_*) e_i e_k^\top C^{j-1-\ell}(\mu, \Gamma_*) \right) \right) \\ &= -\Re \text{tr} \left(U^* \sum_{j=1}^m \sum_{\ell=0}^{j-1} A_j V C^\ell(\mu, \Gamma_*) e_i e_k^\top C^{j-1-\ell}(\mu, \Gamma_*) \right) \\ &= -\Re (e_k^\top G e_i) \quad \text{for } 1 \leq i < k \leq r. \end{aligned}$$

The latter equation follows from the trace identity $\text{tr}(XY) = \text{tr}(YX)$. Analogously we have

$$0 = \frac{\partial \sigma}{\partial \Im \gamma_{ik}}(\Gamma_*) = \Re(-ie_k^\top G e_i) = \Im(e_k^\top G e_i) \quad \text{for } 1 \leq k < i \leq r.$$

Thus, G is upper triangular. Let

$$M = -U^* A_0 V + \sum_{j=1}^m \sum_{\ell=1}^{j-1} C^{j-\ell}(\mu, \Gamma_*) U^* A_j V C^\ell(\mu, \Gamma_*). \quad (11)$$

Then, it is easily verified that

$$\begin{aligned} G C(\mu, \Gamma_*) &= M + U^* \sum_{j=0}^m A_j V C^j(\mu, \Gamma_*) \\ &= M + \sigma(\Gamma_*) U^* U, \end{aligned}$$

where the last equality follows by writing (7) in matrix form. Also,

$$\begin{aligned} C(\mu, \Gamma_*) G &= M + \left(\sum_{j=0}^m C^j(\mu, \Gamma_*) U^* A_j \right) V \\ &= M + \sigma(\Gamma_*) V^* V, \end{aligned}$$

where the last equality follows from (8). Thus,

$$\sigma(\Gamma_*)(U^*U - V^*V) = GC(\mu, \Gamma_*) - C(\mu, \Gamma_*)G. \quad (12)$$

Since G and $C(\mu, \Gamma_*)$ are both upper triangular, the right hand side of this equation is strictly upper triangular. The left hand side is Hermitian. Hence, both sides vanish. Thus, $U^*U = V^*V$.

3.2 Sylvester Equation for Perturbed Matrix Polynomial

In this subsection we show that

$$\dim \left\{ X \in \mathbb{C}^{n \times r} \mid \sum_{j=1}^m A_j X C^j(\mu, \Gamma_*) + (A_0 + \Delta A_*)X = 0 \right\} \geq r \quad (13)$$

under the assumption that V is full rank, where Γ_* is as defined in (6). Our starting point is the singular value equation (7), which could be rewritten as a matrix equation of the form

$$\sum_{j=0}^m A_j V C^j(\mu, \Gamma_*) = \kappa(P, \mu)U$$

Assuming V is full rank we have $V^+V = I$. Consequently,

$$\sum_{j=0}^m A_j V C^j(\mu, \Gamma_*) = \kappa(P, \mu)U V^+V \implies \sum_{j=1}^m A_j V C^j(\mu, \Gamma_*) + (A_0 + \Delta A_*)V = 0.$$

Moreover, consider the subspace of matrices

$$\mathcal{D} := \{D \in \mathbb{C}^{r \times r} \mid C(\mu, \Gamma_*)D - DC(\mu, \Gamma_*) = 0\}$$

commuting with $C(\mu, \Gamma_*)$, which is of dimension at least r (due to Theorem 3.1 in [3]). For all $D \in \mathcal{D}$, we have

$$0 = \sum_{j=1}^m A_j V C^j(\mu, \Gamma_*)D + (A_0 + \Delta A_*)VD = \sum_{j=1}^m A_j (VD) C^j(\mu, \Gamma_*) + (A_0 + \Delta A_*)(VD)$$

meaning each matrix in the set $\{VD \mid D \in \mathcal{D}\}$ is a solution of the Sylvester equation

$$\sum_{j=1}^m A_j X C^j(\mu, \Gamma_*) + (A_0 + \Delta A_*)X = 0.$$

Therefore, we conclude with (13) assuming V is full rank.

3.3 Main Result and Corollaries

Let us first suppose μ consists of distinct scalars. Then all eigenvalues of $C(\mu, \Gamma)$ have algebraic and geometric multiplicities equal to one for all Γ , implying $\mathcal{G}(\mu) = \mathbb{C}^{r(r-1)/2}$. Consequently, we have $\Gamma_* \in \mathcal{G}(\mu)$, where Γ_* is as defined in (6). It follows from Sections 3.1 and 3.2 that $\beta(P, \mu) = \kappa(P, \mu)$ under multiplicity and linear independence assumptions at the optimal Γ_* .

When there are repeated scalars in μ , then there are $\tilde{\mu}$ comprised of distinct scalars and arbitrarily close to μ , where the equality $\beta(P, \tilde{\mu}) = \kappa(P, \tilde{\mu})$ is satisfied under multiplicity and linear independence assumptions. Then, the equality $\beta(P, \mu) = \kappa(P, \mu)$ follows from the continuity of both $\beta(\cdot)$ and $\kappa(\cdot)$ with respect to μ (again under multiplicity and linear independence assumptions). We arrive at the following main result of this paper.

Theorem 3.1 (Distance to Polynomials with Specified Eigenvalues). *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$, $\mathcal{S} := \{\lambda_1, \dots, \lambda_s\}$ be a set of complex scalars, and $r \in \mathbb{Z}^+$.*

(i) *Then the singular value characterization*

$$\beta(P, \mathcal{S}) = \inf_{\mu \in \mathcal{S}^r} \sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \sigma_{n \cdot r - r + 1} \left(\sum_{j=1}^m (C^j(\mu, \Gamma))^T \otimes A_j + I \otimes A_0 \right) \quad (14)$$

holds, for the distance $\beta(P, \mathcal{S})$ as defined in (2), provided that the multiplicity and linear independence assumptions hold at the optimal $\mu \in \mathcal{S}^r$ and $\Gamma \in \mathbb{C}^{r(r-1)/2}$.

(ii) *The minimal ΔA_* in 2-norm such that $\sum_{j=1}^s m_j (P(\lambda) + \Delta A_*) \geq r$ is given by (9), but with μ replaced by μ_* , which is the μ value where the outer infimum in (14) is attained.*

Specifically, when $\mathcal{S} = \{\mu\}$ and $r = 2$, we obtain the following characterization for the distance to the nearest polynomials with μ as a multiple eigenvalue. Even though the details are long and technical, it can indeed be shown that even if the multiplicity and linear independence assumptions are violated, the singular value formula remains valid in this special case.

Corollary 3.2 (Distance to Polynomials with Multiple Eigenvalues). *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$, and $\mu \in \mathbb{C}$. Then*

$$\inf\{\|\Delta A\|_2 \mid P(\lambda) + \Delta A \text{ has } \mu \text{ as a multiple eigenvalue}\} = \sup_{\gamma \in \mathbb{R}} \sigma_{2n-1} \left(\begin{bmatrix} P(\mu) & 0 \\ \gamma P'(\mu) & P(\mu) \end{bmatrix} \right)$$

For the above result concerning the nearest polynomials with a multiple eigenvalue, it is again possible to construct the minimal perturbation using the formula in (9). Also note that the maximization is over the real numbers for this special case, since the singular values of

$$\begin{bmatrix} P(\mu) & 0 \\ \gamma P'(\mu) & P(\mu) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} P(\mu) & 0 \\ |\gamma| P'(\mu) & P(\mu) \end{bmatrix}$$

are the same for all $\gamma \in \mathbb{C}$. Corollary 3.2 could easily be generalized for the nearest polynomials with μ as a multiple eigenvalue of *specified algebraic multiplicity*. Again the optimal perturbations are given by the formula in (9).

Corollary 3.3 (Distance to Polynomials with Eigenvalues of Specified Multiplicity). *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$, $\mu \in \mathbb{C}$ and $r \in \mathbb{Z}^+$. Then*

$$\inf\{\|\Delta A\|_2 \mid P(\lambda) + \Delta A \text{ has } \mu \text{ as an eigenvalue of algebraic multiplicity } \geq r\} =$$

$$\sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \sigma_{n \cdot r - r + 1} \left(\sum_{j=1}^m \left(\begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ \gamma_{21} & \mu & & \vdots \\ & & \ddots & 0 \\ \gamma_{r1} & & \gamma_{r(r-1)} & \mu \end{bmatrix}^j \otimes A_j \right) + I \otimes A_0 \right)$$

provided that the multiplicity and linear independence assumptions hold at the optimal Γ .

Finally in control theory it is of interest to place the eigenvalues of a dynamical system at desired locations in the complex plane. For instance stability is an essential condition. The main result, with $n = r$ and $\mathcal{S} = \mathbb{C}_-$ denoting the open left-half of the complex plane, provides us with a characterization of the nearest stable systems given an unstable higher-order linear system with some of the eigenvalues on the right-half of the complex plane.

Corollary 3.4 (Distance to Stability for Matrix Polynomials). *Let $P(\lambda) := \sum_{j=0}^m \lambda^j A_j$ with $A_j \in \mathbb{C}^{n \times n}$ and such that $\text{rank}(A_m) = n$. Then*

$$\inf\{\|\Delta A\|_2 \mid P(\lambda) + \Delta A \text{ has all eigenvalues with negative real parts}\} =$$

$$\inf_{\mu \in \mathbb{C}_-} \sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \sigma_{n^2 - n + 1} \left(\sum_{j=1}^m \left(\begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ \gamma_{21} & \mu_2 & & \vdots \\ & & \ddots & 0 \\ \gamma_{n1} & & \gamma_{n(n-1)} & \mu_n \end{bmatrix}^j \otimes A_j \right) + I \otimes A_0 \right)$$

provided that the multiplicity and linear independence assumptions hold at the optimal Γ and μ .

4 Numerical Examples

We illustrate our main result Theorem 3.1 and Corollary 3.2 on two examples that can be visualized by means of the ϵ -pseudospectrum of the polynomial $P(\lambda)$. The ϵ -pseudospectrum that is related to our results consists of the eigenvalues of all polynomials within an ϵ neighborhood with respect to the 2-norm and when only the constant coefficient matrix is perturbed, that is

$$\begin{aligned} \Lambda_\epsilon(P) &:= \bigcup_{\|\Delta A\|_2 \leq \epsilon} \Lambda(P(\lambda) + \Delta A) \\ &= \{z \in \mathbb{C} \mid \sigma_n(P(z)) \leq \epsilon\}. \end{aligned}$$

where $\Lambda(P(\lambda))$ denotes the spectrum of the polynomial $P(\lambda)$.

The derivation in the previous section establishes that any stationary point of the inner maximization problem in (14) is a global maximizer as long as the multiplicity and linear

independence assumptions hold. Consequently, we solve the inner problems using quasi-Newton methods numerically. For the numerical solutions of the outer minimization problems we depend on the technique recently described in [2], which exploits the piece-wise analyticity of a singular value function of a matrix function depending on a parameter analytically.

Both of the numerical experiments below is performed on a 5×5 matrix polynomial of degree two, whose entries are selected from a normal distribution with zero mean and unit variance.

4.1 Polynomials with Two Prescribed Eigenvalues

Suppose that $\mathcal{S} = \{\lambda_1, \lambda_2\}$ and $r = 2$ so that two eigenvalues are prescribed, and the distance to a nearest polynomial for which at least two of the eigenvalues belong to \mathcal{S} is sought. Then the singular value formula (14) takes the form

$$\beta(P, \mathcal{S}) = \inf_{\mu \in \mathcal{S}^2} \sup_{\gamma} \sigma_{2n-1} \left(\begin{bmatrix} P(\mu_1) & 0 \\ \gamma \cdot P_{\Delta}(\mu_1, \mu_2) & P(\mu_2) \end{bmatrix} \right) \quad (15)$$

where

$$P_{\Delta}(\mu_1, \mu_2) = \begin{cases} \left(\frac{P(\mu_1) - P(\mu_2)}{\mu_1 - \mu_2} \right) & \text{if } \mu_1 \neq \mu_2, \\ P'(\mu_1) & \text{if } \mu_1 = \mu_2. \end{cases}$$

Here, we calculate this distance for the quadratic matrix polynomial mentioned at the beginning of this section with random entries, and for the prescribed eigenvalues $\mathcal{S} = \{-0.3 + 0.1i, -0.65\}$. The boundaries of the ϵ -pseudospectra of the quadratic matrix polynomial are plotted in Figure 1 together with the prescribed eigenvalues marked by asterisks. In particular the outer curves correspond to the boundary of the ϵ -pseudospectrum for $\epsilon = 0.5879$, which is the computed distance $\beta(P, \mathcal{S})$ by means of the characterization (15). On one of these outer curves one of the prescribed eigenvalues $\lambda_1 = -0.3 + 0.1i$ lies. However, we remark that one of the prescribed eigenvalues does not always have to lie on the boundary of the ϵ -pseudospectrum for $\epsilon = \beta(P, \mathcal{S})$; in general both of the prescribed eigenvalues may lie strictly inside the pseudospectrum.

4.2 Nearest Polynomials with Multiple Eigenvalues

By Corollary 3.2 the distance to a nearest matrix polynomial with a multiple eigenvalue is given by

$$\inf_{\mu \in \mathbb{C}} \sup_{\gamma \in \mathbb{R}} \sigma_{2n-1} \left(\begin{bmatrix} P(\mu) & 0 \\ \gamma P'(\mu) & P(\mu) \end{bmatrix} \right).$$

For a matrix polynomial of size $n \times n$ and degree m the ϵ -pseudospectrum for small ϵ is comprised of nm disjoint components, one around each eigenvalue. The smallest ϵ such that two components of the ϵ -pseudospectrum coalesce is equal to this distance. This is not an obvious fact; indeed for matrices this has been proven by Alam and Bora [1] not long time ago. The extension of the proof due to Alam and Bora for matrix polynomials is straightforward.

For the random quadratic matrix polynomial we compute this distance as 0.3211. Two components of the ϵ -pseudospectrum for $\epsilon = 0.3211$ coalesce as expected in theory. This is illustrated in Figure 2; specifically the inner-most curves represent the boundary of this

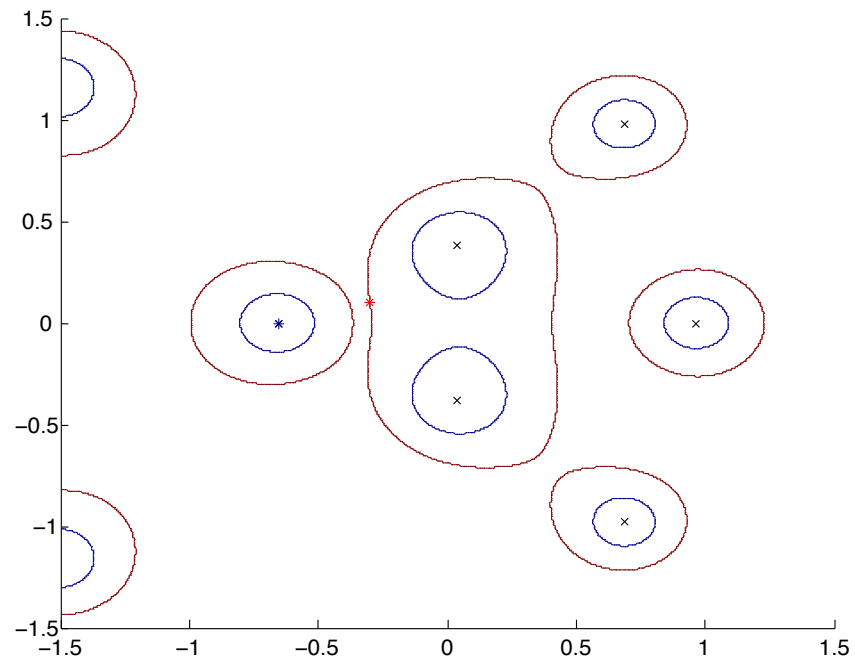


Figure 1: The pseudospectra of the quadratic random matrix polynomial. The asterisks are the prescribed eigenvalues. The black crosses represent the eigenvalues of the matrix polynomial

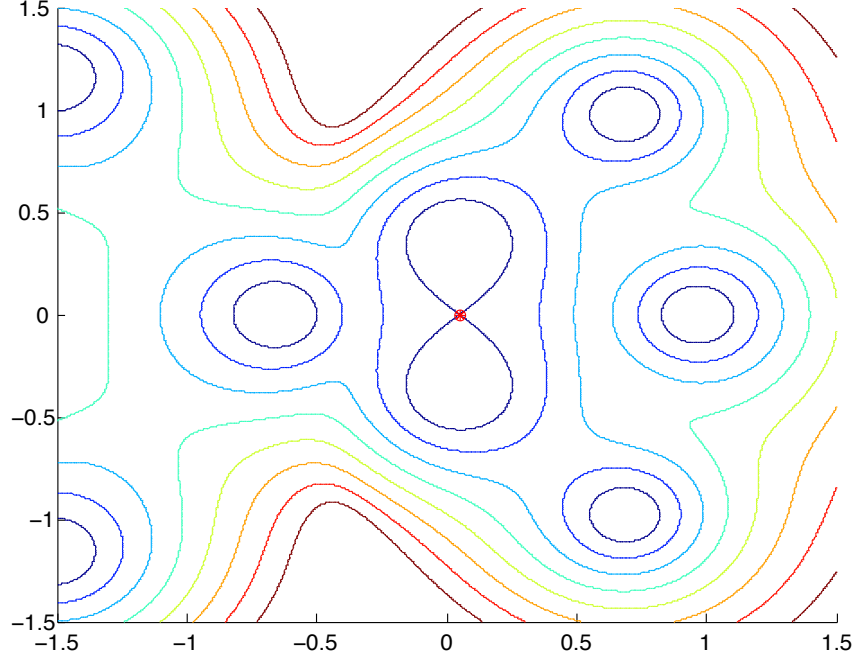


Figure 2: The ϵ -pseudospectra of the quadratic random matrix polynomial for various ϵ are illustrated. The inner most curve corresponds to the boundary of the ϵ -pseudospectrum for ϵ equal to the distance to a nearest matrix with a multiple eigenvalue. The red asterisk is the multiple eigenvalue of a nearest matrix polynomial.

ϵ -pseudospectrum. The point of coalescence of the components $z = 0.0490$, marked by an asterisk, is the multiple eigenvalue of a nearest polynomial.

5 Concluding Remarks

We derived a singular value optimization characterization for the distance from a matrix polynomial to a nearest one with a specified number of eigenvalues belonging to a specified set. We restricted ourselves to square matrix polynomials. Extensions to the rectangular matrix polynomials is straightforward as long as the leading coefficient matrix is full rank.

There are two important open problems that are left untouched by this paper. First, it is more desirable to allow perturbations to all coefficient matrices from an application point of view. In this case an exact singular value formula is not known at the moment. Secondly, the results are proven under mild multiplicity and linear independence assumptions. Our experience with special instances indicates that the singular value formula remains valid, even when these assumptions are not met. We conjecture that neither of the assumptions is needed for the validity of the singular value characterization.

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